My research is in **geometric group theory**, an area of mathematics devoted to studying groups as geometric objects in order to solve algebraic and algorithmic problems, as well as problems in other fields. For example, geometric group theory was used by Agol and Wise to solve Thurston's virtual Haken conjecture, a major open problem in low-dimensional topology [5, 71]; by Sela to solve the famed Tarski conjecture in first-order logic [102]; and by Bridson and Grunewald to solve Grothendieck's First Problem on profinite rigidity [29]. The methods of geometric group theory have also had extraordinary success outside of mathematics, with Carlsson's development of persistent homology in topological data analysis [32], Ghrist's applications of braid groups in robotics [62], and more recently exciting applications of graph braid groups in topological quantum computing by Maciazek et al. [8, 9, 82, 81, 6, 72].

Groups are studied as geometric objects in two ways: through their actions on metric spaces, and by considering groups as metric spaces in their own right via their Cayley graphs. One must beware that a group can have many Cayley graphs, depending on the choice of generating set. However, so long as these generating sets are finite, the distances in any two Cayley graphs are equivalent up to a fixed multiplicative and additive constant; we say the Cayley graphs are **quasi-isometric**. By restricting ourselves to geometric properties that are invariant under quasi-isometry, we therefore obtain invariants of the group itself.

A particularly powerful quasi-isometry invariant is the notion of **hyperbolicity** of a group, first introduced by Gromov in two seminal papers [64, 65]. This property mimics classical hyperbolic geometry in a way that is applicable to any geodesic metric space, including Cayley graphs. Hyperbolic groups enjoy a wide variety of useful properties; for example, they are finitely presented and many algorithmic problems are decidable (often computable as quickly as linear time) [7, 49]. Moreover, hyperbolic groups are "automatic", a formal criterion that implies many calculations can be carried out very efficiently [31]. Examples of hyperbolic groups include finite groups, free groups, small-cancellation groups, and fundamental groups of closed hyperbolic manifolds.

There have been many efforts to generalise Gromov's notion of hyperbolicity in order to apply its rich toolset to broader classes of groups. Generalisations include **relatively hyperbolic groups**, introduced by Gromov and further developed by Farb, Bowditch and many others [64, 52, 28, 36, 94], **acylindrically hyperbolic groups**, introduced by Sela [101] and Bowditch [27] and further developed by Osin [95], and **hierarchically hyperbolic groups** (HHGs), introduced by Behrstock, Hagen, and Sisto [17]. My research interests concern all of these, but I am primarily interested in developing the theory of HHGs.

Hierarchical hyperbolicity is an axiomatisation of geometric features common to a surprisingly large class of seemingly disparate groups, including hyperbolic groups, fundamental groups of the special **cube complexes** used by Agol and Wise, as well as **mapping class groups** of surfaces, which have long been important in low-dimensional topology, and most **3–manifold groups** [17, 18]. Behrstock–Hagen–Sisto's framework unites Masur and Minsky's treatment of mapping class groups via subsurface projections and curve graphs [86, 87, 14, 20] with Haglund and Wise's treatment of cube complexes [71]. Hierarchical hyperbolicity provides powerful tools to simultaneously study all of the groups encompassed, and often allows techniques known for one group to be imported to another group in the class [17, 18, 16, 19]. For example, HHGs satisfy a Masur–Minsky style distance formula and retain many of the strong algorithmic and computational properties of hyperbolic groups [18, 67, 48].

One of my main interests is **quotients of HHGs**. The study of quotients of hyperbolic groups is a classical subject, traditionally approached via small-cancellation theory. Work of Delzant (and independently Ol'shanskii) showed that quotients of hyperbolic groups by high powers of elements with large translation length are hyperbolic [44, 92]. Small-cancellation theory has since been modernised by Osin [93], Wise [109], and Dahmani–Guirardel–Osin [39], who produce similar theories that are applicable to relatively hyperbolic groups, special cube complexes, and mapping class groups, respectively—this was used in the proof the the virtual Haken conjecture [5]. Quotients of 3–manifolds may also be considered via the classical method of Dehn filling [107, 108]. One may therefore wonder whether a version of Delzant's result exists for HHGs. Some progress has been made towards this by Behrstock–Hagen–Martin–Sisto in the case of mapping class groups [15], however I seek to answer this question in its full generality.

I am also interested in **braid groups** of various spaces X. Introduced by Fox–Neuwirth [56], these can be considered as fundamental groups of configuration spaces of X, tracking the motion of a collection of particles as they travel around X. It is a folklore result that the classical braid group of a disc is an HHG; one way to see this is by expressing it as a central extension of a mapping class group [25]. On the other hand, the braid group of a graph is the fundamental group of a special cube complex, hence is an HHG [4, 34]. One may therefore wonder whether all braid groups are HHGs. I seek to answer this question, as well as using HHG structures to characterise other aspects of non-positive curvature in braid groups.

My research programme can be summarised as follows.

(1) **Hierarchically hyperbolic groups.**

(1.1) *Examples of HHGs.* Expand the catalogue of examples of HHGs and apply HHG tools to obtain new results for these groups, including characterisations of (relative) hyperbolicity.

- I have done this for graph braid groups [B2], graph products [B5], and "almost HHGs" [B4]; the latter two with Jacob Russell. In ongoing work with Mark Hagen, we characterise when central extensions of HHGs are HHGs, generalising the fact that classical braid groups are HHGs.

(1.2) *Quotients of HHGs.* Determine when a quotient of an HHG is again an HHG, and determine which other properties are preserved.

- Abbott, Ng, Rasmussen, and I have shown that random quotients of HHGs are asymptotically almost surely HHGs [B1]. Here, our model of randomness is given by taking random walks on the Cayley graph. In ongoing work, we aim to produce a deterministic version of this result. We also aim to show that suitable mapping class group quotients are isomorphic to the automorphism groups of the corresponding curve graph quotients, providing an analogue of Ivanov's famous result that mapping class groups are isomorphic to the automorphism groups of their curve graphs [76].

(2) **Braid groups.**

(2.1) *Graph braid groups.* Characterise (relative) hyperbolicity in graph braid groups, determine necessary and sufficient conditions for a graph braid group to split as a free product, and determine when a graph braid group is isomorphic to a right-angled Artin group.

- I have partially answered all of the above questions [B2, B3]. In ongoing work, I am developing a new HHG structure to fully characterise relative hyperbolicity.

(2.2) *Other braid groups.* Determine which other braid groups are HHGs. Use the HHG structure to prove new results about these braid groups.

- I am currently working on showing that hybrid braid groups, obtained by gluing discs to graphs along cycles, are HHGs. I also aim to answer this question for surface braid groups.

1. Hierarchically hyperbolic groups/spaces (HHGs/HHSs)

Several years ago, Behrstock, Hagen, and Sisto devised a version of non-positive curvature in groups called **hierarchical hyperbolicity**, after identifying a collection of geometric features shared by some surprising groups that were previously thought to be unrelated. Behrstock had proven a number of powerful results for mapping class groups [20, 14] by identifying important properties of Masur and Minsky's hierarchy machinery using subsurface projections and curve graphs [86, 87], while Hagen had developed a curve graph analogue for cube complexes, called the **contact graph**, which among other things allowed geodesics in the cube complex to be tracked via geodesics in the contact graph [68]. Sisto recognised this tracking of geodesics as a version of the **hierarchy paths** found in mapping class groups, an analogy that he had already developed in the context of relatively hyperbolic groups and certain 3–manifold groups, which had allowed him to prove results for these groups reminiscent of Masur–Minsky's and Behrstock's mapping class group results [104, 103, 105]. Drawing inspiration from Kim and Koberda's work [78], which had previously attempted to draw comparisons between right-angled Artin groups and mapping class groups by developing a curve graph analogue, the three authors conspired to develop a general structure called hierarchical hyperbolicity, distilling the properties shared by mapping class groups, groups acting on cube complexes, 3–manifold groups, and relatively hyperbolic groups into a set of axioms [17, 18].

Hierarchical hyperbolicity of a space is a quasi-isometry invariant property which combines elements of both Euclidean and hyperbolic geometry. The geometric information of an HHS X is encoded in a collection of projections onto hyperbolic spaces associated to X, in analogy with the subsurface projections and corresponding curve graphs associated to a mapping class group. These projections are arranged via a partial order called **nesting**, and flats (copies of $\mathbb{Z}^n$) are encoded via a combinatorial relation between the projections called **orthogonality**. Due to the extra structure endowed by the projections and relations, one must be careful to distinguish a hierarchically hyperbolic *space* from a hierarchically hyperbolic *group*. An

HHG is not merely a group whose Cayley graph is an HHS; the hierarchy structure must also be equivariant with respect to the group action. In fact, while the property of being an HHS is quasi-isometry invariant, the property of being an HHG is not [96].

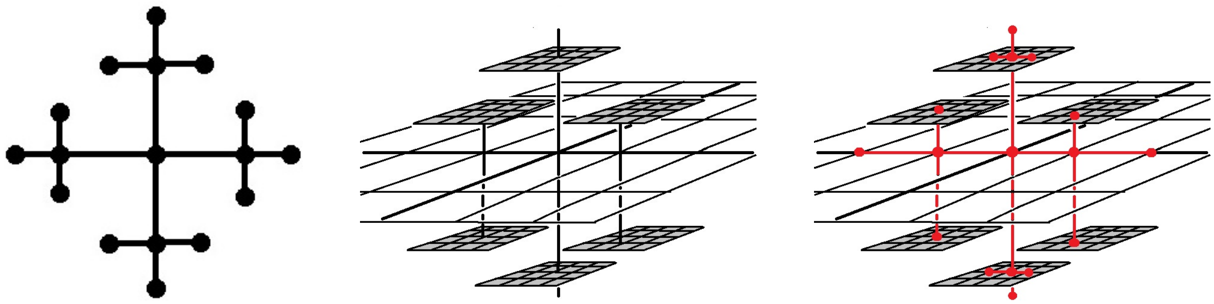

FIGURE 1. The right-angled Artin group $(\mathbb{Z}\times\mathbb{Z})*\mathbb{Z}$ can be endowed with an HHG structure. Its Cayley graph (centre) contains both Euclidean planes and copies of the Cayley graph of the hyperbolic group $\mathbb{Z}*\mathbb{Z}$ (left).

Behrstock, Hagen, and Sisto showed that a wide range of groups/spaces are hierarchically hyperbolic [18], and used this unifying framework to further our understanding of their algebra and geometry [2, 19, 16]. Prominent examples include fundamental groups of Haglund and Wise's compact special cube complexes [71], fundamental groups of closed 3–manifolds with no Nil or Sol components [18], mapping class groups [86, 87, 14, 20], and Teichmüller space with the Teichmüller or Weil–Petersson metric ([97, 46, 50] and [30, 14, 20] respectively). One of the overarching goals of my research is to introduce new classes of groups to the HHG arsenal. So far, I have expanded the catalogue of HHGs by adding graph products (Theorem 1), almost HHGs (Theorem 5), and graph braid groups (Theorem 13) to this list [B4, B5]. These new examples are described in more detail below.

Hierarchically hyperbolic groups have far-reaching applications due to the richness of their structure and the diversity of groups they encompass. For example, all hyperbolic groups are HHGs [19] and all groups that are hyperbolic relative to HHGs are HHGs [18], and moreover these properties are encoded in the HHG structure. It is therefore possible to characterise when a group is hyperbolic or relatively hyperbolic in terms of its HHG structure [19, 99]. Thus, if one is able to construct an HHG structure for a class of groups, one also often obtains characterisations of (relative) hyperbolicity for these groups. So far, I have obtained such characterisations for graph products (Theorem 3) and graph braid groups (Theorem 14 and Section 2.1.3). Other examples of powerful properties possessed by HHGs include a quadratic isoperimetric inequality (and consequently solvable word problem), solvable conjugacy problem, finite asymptotic dimension, a Tits alternative, quasi-isometric rigidity of flats, and a Masur–Minsky style distance formula that allows one to express distances in the group in terms of distances in the projections onto hyperbolic spaces [17, 18, 19, 16, 67].

1.1. **Examples of HHGs.** Here, I detail my results for graph products, almost HHGs, and central extensions of HHGs. For graph braid groups, see Section 2.1.

1.1.1. *Graph products.* Given a finite simplicial graph Γ and a collection of finitely generated groups $\{G_v\}$ indexed by the vertices of Γ, the **graph product** G_Γ is defined to be the free product of the vertex groups $\{G_v\}$ with commutation relations added between elements of G_v and G_w whenever v and w are connected by an edge in Γ. In particular, if the vertex groups are all copies of $\mathbb{Z}$, then G_Γ is the right-angled Artin group (RAAG) with defining graph Γ.

Behrstock, Hagen, and Sisto asked if graph products are HHGs when the vertex groups are HHGs. This was partially answered by Berlai and Robbio, giving a positive answer with some additional requirements on the vertex groups [23]. In joint work with Jacob Russell, we give a complete answer, showing that all graph products of HHGs are themselves HHGs.

Theorem 1 ([B5]). *A graph product of HHGs is an HHG.*

We do this by first showing that graph products G_Γ are **relative HHGs**, that is, they are HHGs modulo the geometry of the vertex groups. This is achieved by replacing the word metric on G_Γ with the syllable metric, a technique originally used by Kim and Koberda in their study of RAAGs [78]. Roughly, this metric can be viewed as the equivalent of the Weil–Petersson metric on Teichmüller space. This new metric gives G_Γ the structure of a **quasi-median graph**, whose geometry has deep analogues with that of cube complexes, as explored by Genevois [58]. This cubical-like geometry allows us to adapt techniques used by Behrstock, Hagen, and Sisto in showing hierarchical hyperbolicity of RAAGs.

We then show that if the vertex groups are themselves HHGs, then we can extend the relative HHG structure through the vertex groups to obtain an HHG structure on G_Γ that uses the word metric. As a byproduct of this method, we answer a second question of Behrstock, Hagen, and Sisto, who asked if graph products are HHSs when endowed with the syllable metric.

Theorem 2 ([B5])**.** *Any graph product with the syllable metric is an HHS.*

As an application of Theorem 1, we are able to characterise when a graph product of hyperbolic groups is itself hyperbolic, recovering a theorem of Meier [88].

Theorem 3 ([B5],[88])**.** *Let Γ be a finite simplicial graph with hyperbolic groups associated to its vertices. Let Γ_F be the subgraph spanned by the vertices associated with finite groups. Then G_Γ is hyperbolic if and only if the following conditions hold.*

(i) There are no edges connecting two vertices of $\Gamma \smallsetminus \Gamma_F$.
(ii) If v is a vertex of $\Gamma \smallsetminus \Gamma_F$ then the link of v is a complete graph.
(iii) Γ_F does not contain any induced squares.

Our use of Genevois' theory of quasi-median graphs provokes more general questions. Quasi-median graphs have analogues of hyperplanes which generalise those of a cube complex, they admit projections onto convex subcomplexes, and moreover these analogues share many of the important geometric properties of their cubical counterparts. Since Behrstock, Hagen, and Sisto show that any cube complex with a **factor system** is an HHS, I conjecture that an equivalent notion exists for quasi-median graphs.

Question 4. Can an analogue of factor systems be developed for quasi-median graphs?

Genevois shows that certain wreath products, diagram products, and graphs of groups can be studied via quasi-median graphs [58], so this could possibly provide a new source of examples of HHGs.

1.1.2. *Almost HHGs.* Abbott, Behrstock, and Durham introduce the class of **almost HHGs** in [2], generalising HHGs by weakening one of the axioms. This axiom is often one of the more awkward ones to prove in practice, and its weaker version is more natural in many ways. In a recent appendix to [2], Russell and I show that this *a priori* weaker structure of an almost HHG can in fact always be promoted to an HHG structure. This simplifies the definition of an HHG by allowing the weaker axiom to be used when proving hierarchical hyperbolicity of a group.

Theorem 5 ([B4])**.** *Every almost HHG admits an HHG structure.*

This theorem strengthens results of Abbott–Behrstock–Durham about almost HHGs by showing that they also hold true in the general setting. In particular, combined with their results, this allows for a complete characterisation of **contracting geodesics** in HHGs. These are geodesics γ that behave like those in hyperbolic space, in the sense that projections of balls onto γ have bounded image, where the bound is independent of the radius of the ball.

Corollary 6 ([2])**.** *Let G be an HHG. For any $D > 0$ and $K \geqslant 1$ there exists $D' > 0$ depending only on D and G such that every (K, K)–quasigeodesic γ has the property that γ is D'–contracting if and only if γ has D–bounded projections onto the hyperbolic spaces in the HHG structure of G.*

Theorem 5 also provides a crucial step in our proof that graph products of HHGs are HHGs (Theorem 1). I expect this result to have many future applications in verifying that a group is hierarchically hyperbolic.

1.1.3. *Central extensions of HHGs.* Consider a central extension

$$1 \to Z \to E \to G \to 1.$$

That is, Z is a subgroup of the centre of E. This defines a cohomology class $\omega \in H^2(G, Z)$, called the **Euler class** of the extension. Work of Gersten shows that if the Euler class is bounded, then E is quasi-isometric to $Z \times G$ [60]. Furthermore, if $Z \cong \mathbb{Z}^n$ then work of Kleiner–Leeb and Neumann–Reeves shows that E is quasi-isometric to $Z \times G$ if and only if the Euler class is weakly bounded [79, 90]. This is further explored by Frigerio and Sisto [57].

If we take G to be an HHG and $Z \cong \mathbb{Z}^n$ (which is also an HHG), then by Theorem 1 and quasi-isometry invariance of being an HHS, we therefore see that the central extension E must be an HHS if the Euler class is weakly bounded. However, one must beware that the property of being an HHG is not always preserved under quasi-isometry [96], so even if E is an HHS, it may not be an HHG. In ongoing work with Mark Hagen, we seek to show that an HHG is produced if and only if the Euler class is bounded.

Conjecture 7. A central extension of an HHG by $\mathbb{Z}^n$ is an HHG if and only if its Euler class is bounded.

Note, if G is a hyperbolic group, then a result of Mineyev tells us that the Euler class is always bounded [89], thus this would imply all central extensions of hyperbolic groups by $\mathbb{Z}^n$ must be HHGs. However, we cannot hope for the Euler class to always be bounded if G is an HHG, since the Heisenberg group provides an example of a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ that is not an HHG.

We aim to prove Conjecture 7 by adapting quasimorphism arguments found in recent papers by Hagen–Martin–Sisto and Hagen–Russell–Sisto–Spriano [69, 70]. Roughly, the idea is that a central extension with bounded Euler class gives rise to a quasimorphism $\phi : E \to \mathbb{Z}^n$. Using results of Abbott–Balasubramanya–Osin [1], this can then be promoted to n actions of E on quasi-lines $L_1, \ldots, L_n$, corresponding to the n generators of $\mathbb{Z}^n$. Combining these into a single action of E on the product $L_1 \times \cdots \times L_n \times \mathrm{Cay}(G, S)$, for a suitable generating set S of G, and showing that this action is metrically proper and cobounded, we then see that E must be an HHG. This last step follows by combining techniques of Behrstock–Hagen–Sisto and Hughes to show that any group acting metrically properly and coboundedly on a product of HHSs and preserving the factors is an HHG [18, 75]. Conversely, we claim that the existence of such an action of E on $L_1 \times \cdots \times L_n \times \mathrm{Cay}(G, S)$ is sufficient to show the Euler class is bounded.

One interesting consequence of this would be that the classical braid group of a disc $B_n(D)$ is an HHG. This is a folklore result that is generally believed to be true, but has not yet been formally written down anywhere.

Corollary 8. *The classical braid group $B_n(D)$ is an HHG.*

This follows from a result of Birman [25] that implies $B_n(D)$ can be expressed as a central extension

$$1 \to Z \to B_n(D) \to G \to 1,$$

where Z is the infinite cyclic group generated by the Garside element, and G is the finite-index subgroup of the mapping class group of the $(n+1)$–times punctured sphere obtained by fixing one of the punctures.

1.2. **Quotients of HHGs.** In an ongoing joint research programme with Carolyn Abbott, Thomas Ng, and Alexander Rasmussen, we study quotients of HHGs. This is a natural evolution of the classical topic of **small-cancellation theory**, dating as far back as Dehn in 1911 [42], which concerns groups whose relations have small overlap with each other. Such groups satisfy a number of nice algebraic, algorithmic, and geometric properties; for example, groups that satisfy sufficiently strong small-cancellation conditions are hyperbolic, and indeed these were one of Gromov's main motivating examples of hyperbolic groups [64]. This was also used by Ol'shanskii to prove that random quotients of the free group are almost surely hyperbolic [91], using Gromov's density model of randomness.

Small-cancellation groups were originally considered in the context of being nice quotients of a free group, however this has since been extended to "small-cancellation quotients" of other groups. One example, due to Delzant, shows that quotients of hyperbolic groups by certain powers of elements with large asymptotic translation length are hyperbolic [44]. In another example, Osin and Groves–Manning consider small-cancellation quotients of relatively hyperbolic groups, showing that such quotients will themselves be relatively hyperbolic [93, 66], with applications including proof of the virtual Haken conjecture [5] and solution of the isomorphism problem for certain relatively hyperbolic groups [38, 41]. This can also be seen as a group-theoretic analogue of Thurston's hyperbolic Dehn surgery theorem, which says that the boundary tori of a cusped hyperbolic 3–manifold can be filled in to produce a closed hyperbolic 3–manifold [108]. In work of Dahmani–Guirardel–Osin, Dahmani, and Dahmani–Hagen–Sisto, another group-theoretic analogue of the Dehn filling procedure is produced for mapping class groups [39, 37, 40], showing among other things that quotients of mapping class groups by high powers of Dehn twists are acylindrically hyperbolic. Yet another generalisation of small-cancellation theory comes in the form of Wise's cubical small-cancellation theory [109], which in particular led to the Malnormal Special Quotient Theorem, a crucial ingredient in the proof of the virtual Haken conjecture [5].

Given all of the above evidence, it is natural to consider whether a version of small-cancellation theory can be developed for HHGs. This topic was first considered by Behrstock–Hagen–Sisto [16], who show that certain quotients of HHGs are HHGs, and later by Behrstock–Hagen–Martin–Sisto [15], who show that quotients of mapping class groups by high powers of Dehn twists (and certain other quotients) are HHGs. We further develop this small-cancellation theory, producing results for random quotients of HHGs and more general results regarding quotients of HHGs by high powers of elements. We also use this to prove an analogue of a famous theorem of Ivanov [76], showing that certain quotients of mapping class groups are isomorphic to the automorphism groups of the corresponding quotients of the curve graph.

1.2.1. *Random quotients.* In an upcoming paper, whose announcement is expected imminently, we show that random quotients of HHGs are asymptotically almost surely HHGs, where our model of randomness is defined by taking random walks on the Cayley graph. This can be seen as a generalisation of Ol'shanskii's result that random quotients of free groups are almost surely hyperbolic [91] and the folklore result that random quotients of hyperbolic groups are asymptotically almost surely hyperbolic.

Theorem 9 ([B1])**.** *Let G be an acylindrical HHG and let $\mu_1, \ldots, \mu_k$ be probability measures on G such that the support of each μ_i is both finite and generates G as a semigroup. Let $\{w_i\}_{i=1}^k$ be k independent random walks of length n and generated by $\mu_1, \ldots, \mu_k$, respectively, and let $m_i = m(\mu_i)$ be the characteristic index of μ_i. Then the quotient*

$$G \big/ \langle\langle w_1^{m_1}, \ldots, w_k^{m_k} \rangle\rangle$$

is an HHG with probability approaching 1 as n tends to infinity.

In fact, in many cases, we do not need to take powers of the random walks, providing stronger results than those known in the deterministic setting.

Corollary 10 ([B1])**.** *Let G be one of the following groups:*

(1) an acylindrically hyperbolic, torsion-free HHG;
(2) a right-angled Artin group that does not split as a direct product;
(3) the mapping class group of a surface of genus g with p punctures;
(4) a right-angled Coxeter group that does not split as a direct product with two infinite factors.

Then the quotient

$$G \big/ \langle\langle w_1, \ldots, w_k \rangle\rangle$$

is an HHG with probability approaching 1 as n tends to infinity.

To prove these results, we first use Dahmani–Guirardel–Osin's **rotating families** machinery [39] to show that, given a collection of hyperbolically embedded subgroups $\{H_i\}_{i\in I}$ in G, if $N_i \trianglelefteq H_i$ such that the family of conjugates $\{N_i^g\}_{i\in I,\, g\in G}$ defines a very rotating family on the nesting-maximal hyperbolic space and the quotients $H_i \big/ N_i$ are hyperbolic, then the quotient $G \big/ \langle\langle \{N_i\} \rangle\rangle$ is an HHG. This generalises work of Behrstock–Hagen–Sisto, who prove this in the single subgroup case [16]. We then apply results of Maher–Tiozzo and Maher–Sisto [84, 83] to show that collections of random walks $\{w_i\}$ satisfy these hypotheses. Here, the hyperbolically embedded subgroups are the unique maximal elementary subgroups $E(w_i)$ containing w_i, and raising w_i to the power of m_i ensures $\langle w_i^{m_i} \rangle$ is normal in $E(w_i)$. The family of conjugates of $w_i^{m_i}$ can then be shown to asymptotically almost surely define a very rotating family by applying results of Dahmani–Guirardel–Osin [39] together with a non-matching argument to show the random walks do not fellow-travel, similar to that used by Abbott and Hull [3].

1.2.2. *General quotients.* We are currently working on showing that versions of the above results also hold in the deterministic setting. We make use of **combinatorial HHGs**, a concept recently introduced by Behrstock, Hagen, Martin, and Sisto which provides a simple combinatorial criterion, in terms of an action on a hyperbolic simplicial complex, for a group to be an HHG [15]. Behrstock, Hagen, Martin, and Sisto apply this to show that quotients of mapping class groups by certain powers of Dehn twists are HHGs. Motivated by a theorem of Delzant, which says that quotients of hyperbolic groups by certain powers of elements with large asymptotic translation length are hyperbolic [44], we conjecture that a more general version of the Behrstock–Hagen–Martin–Sisto result exists.

Conjecture 11. Let G be an acylindrically hyperbolic combinatorial HHG. There exists a constant $N \geqslant 1$ such that for any infinite order element $g \in G$ and any non-zero integer k, the quotient $G/\langle\langle g^{kN} \rangle\rangle$ is an acylindrically hyperbolic combinatorial HHG.

A result of Durham–Hagen–Sisto tells us that any infinite order element g in an HHG G is **axial** [47], that is, there exist associated hyperbolic spaces $C(U_1), \ldots, C(U_n)$ in the HHG structure in which the orbits of $\langle g \rangle$ are quasi-lines. Our idea is to fix a basepoint $x \in G$, consider its projections $\pi_i(x)$ to the hyperbolic spaces $C(U_i)$, and then cone off the axes $\langle g \rangle \cdot \pi_i(x)$ and their conjugates. If we then add in new line domains for each of the axes we coned off, this gives a new combinatorial HHG structure on G. We can then pass to the quotient to obtain a combinatorial HHG structure there.

In practice, the step of passing to the quotient is quite involved, with the Behrstock–Hagen–Martin–Sisto result relying on a generalisation of Dahmani's **composite rotating families** machinery to ensure lifts interact nicely with the combinatorial structure [36]. We expect that this is where most of the hard work will be. Additionally, we expect that we may need to impose a lower bound on the asymptotic translation length of our element g in the hyperbolic spaces $C(U_i)$, in analogy with Delzant's result [44].

By taking successive quotients, we may even be able to obtain a more general result. In the above argument, the domains corresponding to the orbits of $\langle g \rangle$ are arranged to be nesting-minimal in the combinatorial HHG structure. After quotienting by these, we may repeat the same procedure on nesting-minimal domains in the new structure, thus proceeding up the nesting lattice in the original HHG structure. This strategy is executed by Behrstock–Hagen–Martin–Sisto for mapping class groups, to great effect.

Some immediate consequences of our conjecture would be that such quotients have finite asymptotic dimension and solvable word and conjugacy problems [16, 18, 67]. This would be applicable to, for example, quotients of mapping class groups by large powers of reducible elements, or quotients of right-angled Artin groups by parabolic subgroups.

1.2.3. *Curve complex quotients.* A celebrated theorem of Ivanov says that the extended mapping class group $MCG^{\pm}(\Sigma)$ of an orientable surface Σ with genus $g \geqslant 2$ and $n \geqslant 0$ boundary components is isomorphic to the automorphism group $\mathrm{Aut}(C(\Sigma))$ of its curve complex $C(\Sigma)$ [76]. We aim to show that the same theorem can be proven for appropriate quotients of mapping class groups. This would provide a first step towards proving quasi-isometric rigidity results for mapping class group quotients; cf. [73, 20]. This complements recent work by Mangioni–Sisto on the same topic [85].

Conjecture 12. Let Σ be an orientable surface with genus $g \geqslant 2$ and $n \geqslant 0$ boundary components, and let $N \trianglelefteq MCG^{\pm}(\Sigma)$ be such that the family of conjugates $\{N^g\}$ defines a very rotating family on $C(\Sigma)$. Then $MCG^{\pm}(\Sigma) / N \cong \mathrm{Aut}\left(C(\Sigma) / N \right)$.

Our method of proof is to use a result of Aramayona–Leininger that says $C(\Sigma)$ has an exhaustion by **finite rigid sets** [10, 11]; that is, there exists a countable collection of finite subsets X whose union is $C(\Sigma)$ and which have the property that for any locally injective map $\phi : X \hookrightarrow C(\Sigma)$, there exists a unique $f \in \mathrm{Aut}(C(\Sigma))$ with $\phi = f|_X$. Moreover, since $\{N^g\}$ is a very rotating family, results of Dahmani–Guirardel–Osin tell us that N acts on $C(\Sigma)$ with large injectivity radius [39] and so in particular, finite rigid sets are preserved in the quotient. Thus, we can take a finite rigid set $X \subseteq C(\Sigma)$, project to $\bar{X} \subseteq C(\Sigma) / N$, and take $\bar{\psi} \in \mathrm{Aut}\left(C(\Sigma) \big/ N \right)$, which restricts to an injection $\bar{\phi} : \bar{X} \hookrightarrow C(\Sigma) \big/ N$. We then lift to $\phi : X \hookrightarrow C(\Sigma)$ and take the unique extension $f \in \mathrm{Aut}(C(\Sigma))$. Ivanov's theorem then tells us that f corresponds to a unique element $g \in MCG^{\pm}(\Sigma)$, which then gives us an element $\bar{g} \in MCG^{\pm}(\Sigma) \big/ N$.

One may also be able to prove the above conjecture directly using curves on surfaces techniques similar to Ivanov's original proof, by showing that traditional invariants such as intersection number are well-defined in the quotient. Again, this can be shown using large injectivity radius arguments.

2. Braid groups

Given a topological space X, one can construct the **configuration space** $C_n^{top}(X)$ of n particles on X by taking the direct product of n copies of X and removing the diagonal. Informally, this space tracks the movement of the particles through X; removing the diagonal ensures the particles do not collide. One then obtains the *unordered configuration space* $UC_n^{top}(X)$ by taking the quotient by the action of the symmetric group on the coordinates of X^n. Finally, the **braid group** $B_n(X, S)$ is defined to be the fundamental group of $UC_n^{top}(X)$ with base point S (in general we shall assume X to be connected and drop the base point from our notation).

Braid groups have been a popular object of study since they were first introduced by Artin in 1926 [12]. Originally, these were studied geometrically as knots; see Figure 2. One can obtain the configuration space interpretation from this geometric model by taking horizontal cross-sections, each of which gives an arrangement of particles on a disc. Each cross-section can be thought of as a snapshot in time, tracking the locations of the particles as they weave between each other. This configuration space approach was first introduced by Fox and Neuwirth [56].

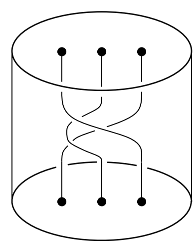

Figure 2. A 3–braid.

Classically, the space X is taken to be a disc D, as in the above example. In this case, Conjecture 7 would imply that $B_n(D)$ is an HHG, as explained in Corollary 8. However, one may also study braid groups of other spaces. Taking X to be a manifold, Birman showed that braid groups are trivial in dimensions 3 and higher [26]. Therefore, in some sense the only interesting cases are when X has dimension 1 or 2; that is, when X is a graph, a surface, or a hybrid of the these. I wish to provide a full analysis of whether such braid groups are HHGs, and use any resulting HHG structure to deduce new properties of $B_n(X)$.

2.1. Graph braid groups.

My recent work has been concerned with the case where X is a finite graph Γ [B2, B3]. These so-called *graph braid groups* were first developed by Abrams [4], who showed that by discretising $UC_n^{top}(\Gamma)$, one can express $B_n(\Gamma)$ as the fundamental group of a non-positively curved cube complex $UC_n(\Gamma)$; this was also shown independently by Świątkowski [106]. Results of Crisp–Wiest, and later Genevois, show that these cube complexes are in fact **special** [34, 59], in the sense of Haglund and Wise [71]. It then follows from work of Behrstock, Hagen, and Sisto that graph braid groups are HHGs [17]. Furthermore, I exploit the cube complex structure to construct an explicit HHG structure on $B_n(\Gamma)$, expressed in terms of combinatorial properties of subgraphs of Γ.

Theorem 13 ([B2]). *$B_n(\Gamma)$ is an HHG for all $n \geqslant 1$.*

One immediate consequence of this is that one can characterise hyperbolicity of $B_n(\Gamma)$ in terms of graph theoretic properties of Γ. This recovers a result of Genevois.

Theorem 14 ([B2],[59]). *$B_n(\Gamma)$ is hyperbolic if and only if one of the following holds.*

(1) $n = 1$.
(2) $n = 2$ *and* Γ *does not contain two disjoint cycles.*
(3) $n = 3$ *and* Γ *does not contain two disjoint cycles, nor does it contain a vertex of valence* $\geqslant 3$ *disjoint from a cycle.*
(4) $n \geqslant 4$ *and* Γ *does not contain two disjoint subgraphs, each of which is a vertex of valence* $\geqslant 3$ *or a cycle.*

Relative hyperbolicity proves to be more elusive. I produce a partial characterisation, but this hinges on a conjecture that remains to be proven. The intricacies of this problem are outlined in more detail in Section 2.1.3.

2.1.1. *Right-angled Artin groups.* As fundamental groups of special cube complexes, graph braid groups $B_n(\Gamma)$ embed in right-angled Artin groups [71]. In fact, Sabalka also shows that all right-angled Artin groups embed in graph braid groups [100]. It is therefore natural to ask when a graph braid group is isomorphic to a right-angled Artin group. This question was first studied by Connolly and Doig in the case where Γ is a linear tree [33], and later by Kim, Ko, and Park, who show that for $n \geqslant 5$, $B_n(\Gamma)$ is isomorphic to a right-angled Artin group if and only if Γ does not contain a subgraph homeomorphic to the letter "A" or a certain tree [77]. I strengthen this theorem by showing that if Γ does not contain a subgraph homeomorphic to the letter "A", then $B_n(\Gamma)$ must split as a non-trivial free product. Thus, for $n \geqslant 5$, $B_n(\Gamma)$ is never isomorphic to a right-angled Artin group with connected defining graph containing at least two vertices.

Theorem 15 ([B3],[77])**.** *Let* Γ *and* Π *be finite connected graphs with at least two vertices and let* $n \geqslant 5$. *Then* $B_n(\Gamma)$ *is not isomorphic to the right-angled Artin group* A_Π.

This theorem raises a natural question: can right-angled Artin groups with connected defining graphs be isomorphic to graph braid groups $B_n(\Gamma)$ for $n \leqslant 4$?

Question 16. Does there exist some $n \geqslant 2$ and a finite connected graph Γ such that $B_n(\Gamma)$ is isomorphic to a non-cyclic right-angled Artin group with connected defining graph?

Most signs point to no. As we shall see below, it is very common for a graph braid group to split as a free product. Moreover, the main examples where it does not split as a free product arise when Γ is non-planar, in which case the abelianisation of $B_n(\Gamma)$ has torsion [77]. However, this also precludes $B_n(\Gamma)$ from being a right-angled Artin group.

2.1.2. *Free products and graphs of groups.* Theorem 15 in fact follows as a consequence of a much more general result that I prove, which provides two criteria for a graph braid group to split as a free product. In the theorem below, a **flower graph** is a graph obtained by gluing cycles and segments along a single central vertex.

Theorem 17 ([B3])**.** *Let* $n \geqslant 2$ *and let* Γ *be a finite graph. Suppose one of the following holds:*

- Γ *is obtained by gluing a non-segment flower graph* Φ *to a connected non-segment graph* Ω *along a vertex* v*, where* v *is either the central vertex of* Φ *or a vertex of* Φ *of valence* 1*;*
- Γ *contains an edge* e *such that* $\Gamma \smallsetminus \mathring{e}$ *is connected but* $\Gamma \smallsetminus e$ *is disconnected, and one of the connected components of* $\Gamma \smallsetminus e$ *is a segment.*

Then $B_n(\Gamma) \cong H * \mathbb{Z}$ *for some non-trivial subgroup* H *of* $B_n(\Gamma)$.

It would be interesting to know if the converse is true; I am not aware of any graph braid groups that split as free products but do not satisfy either of the above criteria.

Question 18. Are there any graph braid groups that split as non-trivial free products but do not satisfy the hypotheses of Theorem 17?

I prove Theorem 17 by applying a technical result on graph of groups decompositions of graph braid groups, which I believe to be of independent interest. Graph of groups decompositions were first considered for pure graph braid groups $PB_n(\Gamma)$ by Abrams [4] and Ghrist [61], and more recently Genevois produced a limited result of this flavour for $B_n(\Gamma)$ [59]. I use the structure of $UC_n(\Gamma)$ as a special cube complex to produce a general construction that allows one to explicitly compute graph of groups decompositions of graph braid groups. In particular, the vertex groups and edge groups are braid groups on proper subgraphs. By iterating this procedure, one is therefore able to express a graph braid group as a combination of simpler, known graph braid groups.

Theorem 19 ([B3])**.** *Let Γ be a finite connected graph and let $e_1, \dots, e_m$ be distinct edges of Γ sharing a common vertex. The graph braid group $B_n(\Gamma)$ decomposes as a graph of groups $(\mathcal{G}, \Lambda)$, where:*

- *$V(\Lambda)$ is the collection of connected components K of $UC_n(\Gamma \smallsetminus (\mathring{e}_1 \cup \dots \cup \mathring{e}_m))$;*
- *$E(\Lambda)$ is the collection of hyperplanes H of $UC_n(\Gamma)$ labelled by some e_i, where H joins K and L if it has one combinatorial hyperplane in K and another in L;*
- *for each $K \in V(\Lambda)$, we have $G_K = B_n(\Gamma \smallsetminus (\mathring{e}_1 \cup \dots \cup \mathring{e}_m), S_K)$ for some $S_K \in K$;*
- *for each $H_i \in E(\Lambda)$ labelled by e_i, we have $G_{H_i} = B_{n-1}(\Gamma \smallsetminus e_i, S_{H_i} \cap (\Gamma \smallsetminus e_i))$, for some S_{H_i} in one of the combinatorial hyperplanes $H_i^{\pm}$;*
- *for each edge $H \in E(\Lambda)$ joining vertices $K, L \in V(\Lambda)$, the monomorphisms $\phi_H^{\pm}$ are induced by the inclusion maps of the combinatorial hyperplanes $H^{\pm}$ into K and L.*

By selecting the edges $e_1, \dots, e_m$ carefully, one may often be able to arrange for the edge groups of $(\mathcal{G}, \Lambda)$ to be trivial. This results in a wide range of graph braid groups that split as non-trivial free products, as shown in Theorem 17.

This construction also aids in the computation of specific graph braid groups, especially when combined with results in which I give combinatorial criteria for adjacency of two vertices of Λ, as well as providing a way of counting how many edges connect each pair of vertices of Λ. Traditionally, graph braid group computations are performed by using Farley and Sabalka's discrete Morse theory to find explicit presentations [54, 55], however in practice these presentations are often highly complex, difficult to compute, and are obfuscated by redundant generators and relators. I believe the graph of groups approach to be both easier to apply and more powerful in many situations, as evidenced by my theorems and the numerous examples I am able to compute [B3].

2.1.3. *(Relative) hyperbolicity.* The graphical requirements in the cases of $n = 3$ and $n \geqslant 4$ in Theorem 14 turn out to be quite restrictive: $B_3(\Gamma)$ is hyperbolic if and only if Γ is a tree, a sun graph, a flower graph, or a pulsar graph; and for $n \geqslant 4$, $B_n(\Gamma)$ is hyperbolic if and only if Γ is a flower graph. One shortcoming of this theorem was that it introduced classes of graphs whose braid groups were unknown: sun graphs and pulsar graphs (braid groups on flower graphs are known to be free [59], while braid groups on trees are known to be free for $n = 3$ [54]). By applying Theorem 17, I am able to (partially) answer a question of Genevois [59] and thus provide a more complete algebraic characterisation of hyperbolicity. The only exception is when Γ is a **generalised theta graph**, which proves resistant to computation. Here, a generalised theta graph Θ_m is a graph obtained by gluing m cycles along a non-trivial segment.

Theorem 20 ([B3],[59])**.** *Let Γ be a finite connected graph that is not homeomorphic to Θ_m for any $m \geqslant 0$. The braid group $B_3(\Gamma)$ is hyperbolic only if $B_3(\Gamma) \cong H * \mathbb{Z}$ for some group H.*

Genevois also provides a graphical characterisation of toral relative hyperbolicity [59]. Again, this theorem introduced several classes of graphs for which the braid groups were unknown. In the case of $n = 4$, this is a finite collection of graphs: the graphs homeomorphic to the letters "H", "A", and "θ". I am able to precisely compute the braid groups of these graphs, completing the algebraic characterisation of toral relative hyperbolicity for $n = 4$ and answering a question of Genevois [59].

Theorem 21 ([B3],[59])**.** *Let Γ be a finite connected graph. The braid group $B_4(\Gamma)$ is toral relatively hyperbolic only if it is either a free group or isomorphic to $F_{10} * \mathbb{Z}^2$ or $F_5 * \mathbb{Z}^2$ or an HNN extension of $\mathbb{Z} * \mathbb{Z}^2$.*

It is much more difficult to characterise relative hyperbolicity in general for graph braid groups. Indeed, I show that in some sense it is impossible to obtain a graphical characterisation of the form that exists for right-angled Artin and Coxeter groups, by providing an example of a graph braid group that is relatively hyperbolic but is not hyperbolic relative to any braid groups of proper subgraphs. This answers a question of Genevois in the negative [59].

Theorem 22 ([B3]). *There exists a graph braid group $B_n(\Gamma)$ that is hyperbolic relative to a proper, non-relatively hyperbolic subgroup P that is not contained in any graph braid group of the form $B_k(\Lambda)$ for $k \leqslant n$ and $\Lambda \subsetneq \Gamma$. See Figure 3.*

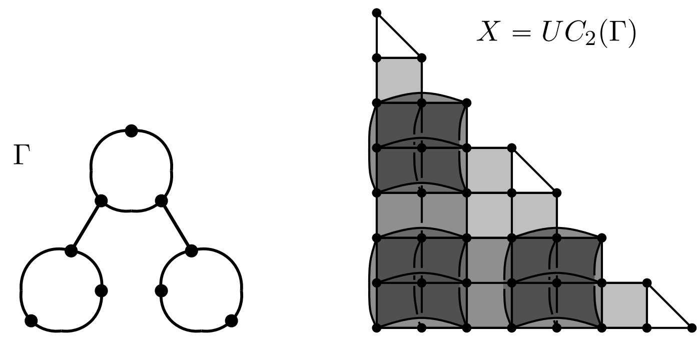

FIGURE 3. $B_2(\Gamma)$ is isomorphic to $\pi_1(X)$, which is hyperbolic relative to the subgroup P generated by the fundamental groups of the three shaded tori.

Note that the peripheral subgroup P in the above example is precisely the group constructed by Croke and Kleiner in [35]. In particular, it is isomorphic to the right-angled Artin group A_Π where Π is a segment of length 3. This theorem indicates that non-relatively hyperbolic behaviour cannot be localised to specific regions of the graph Γ. Instead, non-relative hyperbolicity is in some sense a property intrinsic to the special cube complex structure.

Unfortunately, this means the HHG structure constructed in my Ph.D. thesis [B2] is insufficient to capture the relatively hyperbolic structure. Russell shows that HHGs are relatively hyperbolic if and only if they admit an HHG structure $\mathfrak{S}$ with **isolated orthogonality** [99], a simple criterion restricting where orthogonality may occur in the HHG structure. Roughly, this means there exist $I_1, \ldots, I_n \in \mathfrak{S}$ that correspond to the peripheral subgroups. However, the above example is problematic here; since the peripheral subgroup P does not arise as a braid group on a subgraph of Γ, it does not appear in the natural HHG structure that I describe [24]. Thus, Russell's isolated orthogonality criterion is not able to detect relative hyperbolicity of $B_2(\Gamma)$ through this specific HHG structure. The question of how to classify relative hyperbolicity in graph braid groups therefore still remains.

Question 23. When is a graph braid group relatively hyperbolic?

One potential approach would be to construct a hierarchically hyperbolic structure on $B_n(\Gamma)$ with greater granularity than the existing one, and then apply Russell's isolated orthogonality criterion. In particular, one would have to construct a factor system on the cube complex $UC_n(\Gamma)$ that contains enough subcomplexes to guarantee that any peripheral subgroup of $B_n(\Gamma)$ will always appear as the fundamental group of such a subcomplex. Indeed, I prove a characterisation of relative hyperbolicity in graph braid groups under the assumption that such an HHG structure exists [B2], which proceeds by a similar argument to Levcovitz's characterisation of relative hyperbolicity in right-angled Coxeter groups [80].

2.2. **Other braid groups.** Having considered the case of braid groups in dimension 1, it remains to study dimension 2. For example, one may study braid groups of surfaces, or braid groups of hybrid spaces consisting of combinations of graph and surfaces.

2.2.1. *Surface braid groups.* As described in Corollary 8, it is a folklore result that the braid group of a disc is an HHG. However, the question remains open for surfaces in general.

Question 24. If Σ is an orientable surface, is the braid group $B_n(\Sigma)$ an HHG?

The fact that disc braid groups and graph braid groups are HHGs already provides some evidence for a positive answer to this question. Beyond this, one may also note that the word problem has been shown to be solvable for surface braid groups by González-Meneses [63], as has the conjugacy problem in one specific case, by Bellingeri–Godelle (although it remains open in general) [22]. These are both known to be solvable for HHGs [18, 67].

One may also look to a theorem of Birman that gives an exact sequence relating surface braid groups to mapping class groups [25]. In the case that Σ has genus $g \geqslant 2$, we have the following short exact sequence:

$$1 \to B_n(\Sigma) \to MCG^{\pm}(\Sigma; \{p_1, \ldots, p_n\}) \to MCG^{\pm}(\Sigma) \to 1.$$

Here, $p_1, \ldots, p_n$ are marked points on Σ corresponding to the n particles in the braid group. Thus, $B_n(\Sigma)$ is isomorphic to the kernel of the homomorphism given by forgetting these marked points. One must be careful with this approach; the kernel is known to be highly distorted, thus it may not be possible to recover an HHG structure on $B_n(\Sigma)$ directly from the ambient structure of the mapping class group.

However, this approach may be fruitful for obtaining HHG structures on $B_n(\Sigma)$–extensions of subgroups of $MCG^{\pm}(\Sigma)$. This is a direct generalisation of the $n = 1$ case (note that $B_1(\Sigma) = \pi_1(\Sigma)$). In the case of $n = 1$, Farb and Mosher [53] show that any group homomorphism $G \to MCG^{\pm}(\Sigma)$ gives rise to a group E_G and a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & E_G & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & MCG^{\pm}(\Sigma; p) & \longrightarrow & MCG^{\pm}(\Sigma) & \longrightarrow & 1
\end{array}$$

and conversely, every group extension $1 \to \pi_1(\Sigma) \to E \to G \to 1$ determines a homomorphism $G \to MCG^{\pm}(\Sigma)$, which in turn determines an extension $1 \to \pi_1(\Sigma) \to E_G \to G \to 1$ isomorphic to the given extension. This follows from the fact that the outer automorphism group $\mathrm{Out}(\pi_1(\Sigma))$ is isomorphic to $MCG^{\pm}(\Sigma)$, a result due to Dehn–Nielsen [43] and Baer [13]. Thus, the $\pi_1(\Sigma)$–extensions of a group G are in bijective correspondence with the monodromy homomorphisms $G \to \mathrm{Out}(\pi_1(\Sigma)) \cong MCG^{\pm}(\Sigma)$.

The isomorphism $\mathrm{Out}(B_n(\Sigma)) \cong MCG^{\pm}(\Sigma)$ in fact holds for any $n \neq 2$; this is a result of Bellingeri [21]. Thus, one may replace the short exact sequences in the above commutative diagram with the more general versions, giving a bijective correspondence between the $B_n(\Sigma)$–extensions of a group G and the monodromy homomorphisms $G \to \mathrm{Out}(B_n(\Sigma)) \cong MCG^{\pm}(\Sigma)$.

Motivated by analogies between $\mathbb{H}^n$ and the Teichmüller space $\mathcal{T}(\Sigma)$ of a surface, Farb and Mosher adapted the notion of convex cocompact Kleinian groups to define a subgroup $G \leqslant MCG^{\pm}(\Sigma)$ to be **convex cocompact** if it acts cocompactly on a quasi-convex subset of $\mathcal{T}(\Sigma)$. Work of Farb–Mosher [53] and Hamenstädt [74] then shows that a $\pi_1(\Sigma)$–extension E of a group G is hyperbolic if and only if the associated homomorphism $G \to MCG^{\pm}(\Sigma)$ has finite kernel and convex cocompact image. We may ask the same question for $B_n(\Sigma)$–extensions in general.

Question 25. Let $G \to MCG^{\pm}(\Sigma)$ be a group homomorphism with finite kernel and convex cocompact image. Is the corresponding $B_n(\Sigma)$–extension hierarchically hyperbolic?

Note that one cannot hope for such $B_n(\Sigma)$–extensions to be hyperbolic, since $B_n(\Sigma)$ contains a $\mathbb{Z}^2$ subgroup for $n \geqslant 2$. However, evidence of hierarchical hyperbolicity comes from the ongoing programme to study candidates for **"geometrically finite"** subgroups $G \leqslant MCG^{\pm}(\Sigma)$, whose $\pi_1(\Sigma)$–extensions are conjectured to be HHGs by Dowdall–Durham–Leininger–Sisto [45]. These subgroups generalise the convex cocompact ones, and some results in this direction have already been proven by Dowdall–Durham–Leininger–Sisto and Russell [45, 98]. Given a positive answer to Question 24, one may therefore hope for a positive answer to Question 25, too.

In fact, a special case of the Fadell–Neuwirth short exact sequence for pure surface braid groups [51] may help illuminate Question 24, too:

$$1 \to \pi_1(\Sigma \setminus \{p_1, \ldots, p_{n-1}\}) \to PB_n(\Sigma) \to PB_{n-1}(\Sigma) \to 1.$$

By inducting on n, one may use this short exact sequence to express $PB_n(\Sigma)$ as a series of surface group extensions, beginning with an expression of $PB_2(\Sigma)$ as a $\pi_1(\Sigma \setminus p)$–extension of $\pi_1(\Sigma)$. Thus, if we can show that this fits into the framework of the Dowdall–Durham–Leininger–Sisto conjecture, this would give a way of showing that (pure) surface braid groups are HHGs.

A positive answer to Question 24 would prove several new properties for surface braid groups. For example, it would show that they have a quadratic isoperimetric inequality, that they satisfy a Tits alternative, and that they have solvable conjugacy problem (the latter is a well-known open question for surface braid groups [22]).

2.2.2. *Hybrid braid groups.* One may also consider braid groups $B_n(X)$ where X is obtained by gluing graphs and surfaces together. Since graph braid groups are known to be HHGs via their structure as fundamental groups of special cube complexes (Theorem 13) and disc braid groups are known to be HHGs via their structure as extensions of mapping class groups (Corollary 8), one expects $B_n(X)$ to be an HHG with some kind of hybrid structure arising by combining the cube complex structure with the mapping class group structure. This question was originally proposed to me by Genevois.

Question 26. Let X be a space obtained by gluing discs to a finite graph Γ, where the discs are glued to cycles of Γ along their boundary. What are the properties of $B_n(X)$? Is it an HHG?

If the above cycles (call them C_i) each contain at most one vertex v_i of valence $\geqslant 3$, then I claim that a version of Theorem 19 can be applied, where the edges $e_1, \ldots, e_m$ are the edges of Γ sharing the vertex v_i and not contained in C_i. These edges collectively separate the disc D_i from the rest of X, thus the vertex and edge groups are products of braid groups of D_i and braid groups of $X \smallsetminus D_i$. Applying this to each disc, we can therefore express $B_n(X)$ as a graph of groups where the vertex and edge groups are products of graph braid groups and disc braid groups.

Conjecture 27. Let X be a space obtained by gluing discs $D_1, \ldots, D_k$ to disjoint cycles $C_1, \ldots, C_k$ of a graph Γ, where each C_i has at most one vertex of valence $\geqslant 3$. Then $B_n(X)$ decomposes as a graph of groups whose vertex groups and edge groups are products graph braid groups and disc braid groups.

Having proven this conjecture, I will apply Theorem 1 together with Theorem 13 and Corollary 8 to show that the vertex and edge groups are HHGs. A theorem of Berlai and Robbio [23] regarding graphs of HHGs can then be applied, completing the proof that $B_n(X)$ is an HHG.

Note that if we remove the conditions on the cycles C_i, then we simply obtain the definition of a 2–dimensional CW-complex. Thus, one may study hybrid braid groups in their full generality as braid groups of 2–complexes. This allows us to apply a discretisation process as in the surface braid group case. In the best case scenario, our techniques for surface braid groups may generalise completely.

Question 28. Let X be a 2–dimensional CW-complex. Is $B_n(X)$ an HHG?

Publications and preprints by Daniel Berlyne

[B1] C. Abbott, D. Berlyne, T. Ng, and A. Rasmussen. Random quotients of hierarchically hyperbolic groups. *In preparation.* When ready, it will be available at `http://www.berlyne.net/assets/random_quotients.pdf/`.

[B2] D. Berlyne. Hierarchical hyperbolicity of graph products and graph braid groups. *CUNY Academic Works*, 2021.

[B3] D. Berlyne. Graph of groups decompositions of graph braid groups. *arxiv:2209.03860*, 2022.

[B4] D. Berlyne and J. Russell. *Appendix to* Largest acylindrical actions and stability in hierarchically hyperbolic groups. *Trans. Amer. Math. Soc. Ser. B*, 8:94–104, 2021. Primary article by C. Abbott, J. Behrstock, and M. G. Durham.

[B5] D. Berlyne and J. Russell. Hierarchical hyperbolicity of graph products. *Groups Geom. Dyn.*, 16(2):523–580, 2022.

Other cited publications and preprints

[1] C. Abbott, S. Balasubramanya, and D. Osin. Hyperbolic structures on groups. *Algebr. Geom. Topol.*, 19:1747–1835, 2019.

[2] C. Abbott, J. Behrstock, and M. G. Durham. Largest acylindrical actions and stability in hierarchically hyperbolic groups. With an appendix by D. Berlyne and J. Russell. *Trans. Amer. Math. Soc. Ser. B*, 8:66–104, 2021.

[3] C. Abbott and M. Hull. Random walks and quasi-convexity in acylindrically hyperbolic groups. *J. Topol.*, 14(3):992–1026, 2021.

[4] A. Abrams. Configuration spaces and braid groups of graphs. *PhD thesis, University of California, Berkeley*, 2000.

[5] I. Agol. The virtual haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With appendix by Ian Agol, Daniel Groves, and Jason Manning.

[6] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. Fisher. Non-abelian statistics and topological quantum information processing in 1D wire networks. *Nat. Phys.*, 7(5):412–417, 2011.

[7] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short. Notes on word-hyperbolic groups. In *E. Ghys, A. Haefliger, & A. Verjovsky (Eds.)*, Proceedings of the conference “Group Theory from a Geometric Viewpoint”, pages 2–63. World Scientific, 1991.

[8] B. H. An and T. Maciazek. Universal properties of anyon braiding on one-dimensional wire networks. *Phys. Rev. B*, 102(20):201407(R), 2020.
[9] B. H. An and T. Maciazek. Geometric presentations of braid groups for particles on a graph. *Comm. Math. Phys.*, 384:1109–1140, 2021.
[10] J. Aramayona and C. Leininger. Finite rigid sets in curve complexes. *J. Topol. Anal.*, 5(2):183–203, 2013.
[11] J. Aramayona and C. Leininger. Exhausting curve complexes by finite rigid sets. *Pac. J. Math.*, 282(2):257–283, 2016.
[12] E. Artin. Theorie der Zöpfe. *Abh. Math. Sem. Hamburg*, 4:47–72, 1926.
[13] R. Baer. Isotopien von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen. *J. Reine Angew. Math.*, 159:101–116, 1928.
[14] J. Behrstock. Asymptotic geometry of the mapping class group and Teichmüller space. *Geom. Topol.*, 10:1523–1578, 2006.
[15] J. Behrstock, M. Hagen, A. Martin, and A. Sisto. A combinatorial take on hierarchical hyperbolicity and applications to quotients of mapping class groups. *arXiv:2005.00567*, 2020.
[16] J. Behrstock, M. F. Hagen, and A. Sisto. Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups. *Proc. Lond. Math. Soc. (3)*, 114(5):890–926, 2017.
[17] J. Behrstock, M. F. Hagen, and A. Sisto. Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups. *Geom. Topol.*, 21(3):1731–1804, 2017.
[18] J. Behrstock, M. F. Hagen, and A. Sisto. Hierarchically hyperbolic spaces II: Combination theorems and the distance formula. *Pacific J. Math.*, 299:257–338, 2019.
[19] J. Behrstock, M. F. Hagen, and A. Sisto. Quasiflats in hierarchically hyperbolic spaces. *Duke Math. J.*, 170(5):909–996, 2021.
[20] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher. Geometry and rigidity of mapping class groups. *Geom. Topol.*, 16(2):781–888, 2012.
[21] P. Bellingeri. On automorphisms of surface braid groups. *J. Knot Theory Ramif.*, 17(1):1–11, 2008.
[22] P. Bellingeri and E. Godelle. Questions on surface braid groups. *hal-00004548*, 2005.
[23] F. Berlai and B. Robbio. A refined combination theorem for hierarchically hyperbolic groups. *Groups Geom. Dyn.*, 14(4):1127–1203, 2020.
[24] D. Berlyne. Hierarchical hyperbolicity of graph products and graph braid groups. *CUNY Academic Works*, 2021.
[25] J. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.
[26] J. Birman. On braid groups. *Comm. Pure Appl. Math.*, 22:41–72, 1969.
[27] B. Bowditch. Tight geodesics in the curve complex. *Invent. Math.*, 171:281–300, 2008.
[28] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.
[29] M. Bridson and F. Grunewald. Grothendieck's problems concerning profinite completions and representations of groups. *Ann. Math.*, 160(1):359–373, 2004.
[30] J. F. Brock. The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores. *J. Amer. Math. Soc.*, 16(3):495–535, 2003.
[31] J. Cannon. The combinatorial structure of cocompact discrete hyperbolic groups. *Geom. Dedicata*, 16:123–148, 1984.
[32] G. Carlsson. Topology and data. *Bull. Am. Math. Soc.*, 46(2):255–308, 2009.
[33] F. Connolly and M. Doig. On braid groups and right-angled Artin groups. *Geom. Dedicata*, 172:179–190, 2014.
[34] J. Crisp and B. Wiest. Embeddings of graph braid and surface groups in right-angled artin groups and braid groups. *Algebr. Geom. Topol.*, 4(1):439–472, 2004.
[35] C. Croke and B. Kleiner. Spaces with nonpositive curvature and their ideal boundaries. *Topology*, 39(3):549–556, 2000.
[36] F. Dahmani. Les groupes relativement hyperboliques et leurs bords. *PhD thesis, University of Strasbourg*, 2003.
[37] F. Dahmani. The normal closure of big Dehn twists, and plate spinning with rotating families. *Geom. Topol.*, 22:4113–4144, 2018.
[38] F. Dahmani and V. Guirardel. Recognizing a relatively hyperbolic group by its Dehn fillings. *Duke Math. J.*, 167(12):2189–2241, 2018.
[39] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Memoirs of the American Mathematical Society*, 245(1156), 2017.
[40] F. Dahmani, M. Hagen, and A. Sisto. Dehn filling Dehn twists. *Proc. R. Soc. Edinb. A: Math.*, 151(1):28–51, 2021.
[41] F. Dahmani and N. Touikan. Deciding isomorphy using Dehn fillings: the splitting case. *Invent. Math.*, 215:81–169, 2019.
[42] M. Dehn. Über unendliche diskontinuierliche Gruppen. *Math. Ann.*, 71:116–144, 1911.
[43] M. Dehn and J. Nielsen. Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. *Acta Math.*, 50:189–358, 1927.
[44] T. Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Math. J.*, 83(3):661–682, 1996.
[45] S. Dowdall, M. Durham, C. Leininger, and A. Sisto. Extensions of Veech groups II: Hierarchical hyperbolicity and quasi-isometric rigidity. *arXiv:2111.00685*, 2021.
[46] M. G. Durham. The augmented marking complex of a surface. *Journal of the London Mathematical Society*, 94(3):933, 2016.
[47] M. G. Durham, M. F. Hagen, and A. Sisto. Boundaries and automorphisms of hierarchically hyperbolic spaces. *Geom. Topol.*, 21(6):3659–3758, 2017.
[48] M. G. Durham, Y. Minsky, and A. Sisto. Stable cubulations, bicombings, and barycenters. *arXiv:2009.13647*, 2020.

[49] D. Epstein and D. Holt. The linearity of the conjugacy problem in word-hyperbolic groups. *Int. J. Alg. Comput.*, 16(2):287–305, 2006.
[50] A. Eskin, H. A. Masur, and K. Rafi. Large-scale rank of Teichmüller space. *Duke Math. J.*, 166(8):1517–1572, 2017.
[51] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.
[52] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.
[53] B. Farb and L. Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol.*, 6:91–152, 2002.
[54] D. Farley and L. Sabalka. Discrete Morse theory and graph braid groups. *Algebr. Geom. Topol.*, 5(3):1075–1109, 2005.
[55] D. Farley and L. Sabalka. Presentations of graph braid groups. *Forum Math.*, 24(4):827–859, 2012.
[56] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962.
[57] R. Frigerio and A. Sisto. Central extensions and bounded cohomology. *arXiv:2003.01146*, 2020.
[58] A. Genevois. Cubical-like geometry of quasi-median graphs and applications to geometric group theory. *Ph.D. thesis, Université d'Aix-Marseille*, 2017.
[59] A. Genevois. Negative curvature in graph braid groups. *Int. J. Algebra Comput.*, 31(1):81–116, 2021.
[60] S. Gersten. Bounded cocycles and combings of groups. *Internat. J. Algebra Comput.*, 2:307–326, 1992.
[61] R. Ghrist. Configuration spaces and braid groups on graphs in robotics, in "Knots, Braids, and Mapping Class Groups: Papers Dedicated to Joan S. Birman (New York, 1998)". *AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI*, 24:29–40, 2001.
[62] R. Ghrist. Configuration spaces, braids, and robotics. *Lecture Note Series, Inst. Math. Sci., NUS*, 19:263–304, 2010.
[63] J. González-Meneses. New presentations of surface braid groups. *J. Knot Theory Ramif.*, 10:431–451, 2001.
[64] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
[65] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.
[66] D. Groves and J. F. Manning. Dehn filling in relatively hyperbolic groups. *Israel J. Math.*, 168:317–429, 2008.
[67] T. Haettel, N. Hoda, and H. Petyt. The coarse Helly property, hierarchical hyperbolicity, and semihyperbolicity. *arXiv:2009.14053*, 2020.
[68] M. Hagen. Weak hyperbolicity of cube complexes and quasi-arboreal groups. *J. Topol.*, 7(2):385–418, 2014.
[69] M. Hagen, A. Martin, and A. Sisto. Extra-large Artin groups are hierarchically hyperbolic. *arXiv:2109.04387*, 2021.
[70] M. Hagen, J. Russell, A. Sisto, and D. Spriano. Equivariant hierarchically hyperbolic structures for 3–manifold groups via quasimorphisms. *arXiv:2206.12244*, 2022.
[71] F. Haglund and D. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.
[72] B. Halperin, Y. Oreg, A. Stern, G. Refael, J. Alicea, and F. von Oppen. Adiabatic manipulations of Majorana fermions in a three-dimensional network of quantum wires. *Phys. Rev. B*, 85(14):144501, 2001.
[73] U. Hamenstädt. Geometry of the mapping class groups III: Quasi-isometric rigidity. *arXiv:0512429*, 2005.
[74] U. Hamenstädt. Word hyperbolic extensions of surface groups. *arXiv:0505244*, 2005.
[75] S. Hughes. Lattices in a product of trees, hierarchically hyperbolic groups, and virtual torsion-freeness. *Bull. London Math. Soc.*, 54(4):1413–1419, 2022.
[76] N. Ivanov. Automorphisms of complexes of curves and of Teichmüller spaces. *Int. Math. Res. Not.*, 1997(14):651–666, 1997.
[77] J. H. Kim, K. H. Ko, and H. W. Park. Graph braid groups and right-angled Artin groups. *Trans. Amer. Math. Soc.*, 364(1):309–360, 2012.
[78] S.-H. Kim and T. Koberda. The geometry of the curve graph of a right-angled Artin group. *Internat. J. Algebra Comput.*, 24(2):121–169, 2014.
[79] B. Kleiner and B. Leeb. Groups quasi-isometric to symmetric spaces. *Comm. Anal. Geom.*, 9:239–260, 2001.
[80] I. Levcovitz. Characterizing divergence and thickness in right-angled Coxeter groups. *arXiv:2007.13796*, 2020.
[81] T. Maciazek, A. Conlon, G. Vercleyen, and J. Slingerland. Extending the planar theory of anyons to quantum wire networks. *arXiv:2301.06590*, 2023.
[82] T. Maciazek and A. Sawicki. Non-abelian quantum statistics on graphs. *Comm. Math. Phys.*, 371:921–973, 2019.
[83] J. Maher and A. Sisto. Random subgroups of acylindrically hyperbolic groups and hyperbolic embeddings. *Int. Math. Res. Not. IMRN*, (13):3941–3980, 2019.
[84] J. Maher and G. Tiozzo. Random walks, WPD actions, and the Cremona group. *Proc. Lond. Math. Soc.*, 123(2):153–202, 2021.
[85] G. Mangioni and A. Sisto. Rigidity of mapping class groups mod powers of twists. *arXiv:2212.11014*, 24(2), 2022.
[86] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.
[87] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.
[88] J. Meier. When is the graph product of hyperbolic groups hyperbolic? *Geom Dedicata*, 61(1):29–41, 1996.
[89] I. Mineyev. Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.*, 11:807–839, 2001.
[90] W. Neumann and L. Reeves. Regular cocycles and biautomatic structures. *Internat. J. Algebra Comput.*, 6:313–324, 1996.
[91] A. Ol'shanskii. Almost every group is hyperbolic. *International Journal of Algebra and Computation*, 2(1):1–17, 1992.
[92] A. Ol'shanskii. On residualing homomorphisms and G-subgroups of hyperbolic groups. *Int. J. Algebra Comput.*, 3(4):365–409, 1993.

[93] D. Osin. Peripheral fillings of relatively hyperbolic groups. *Invent. Math.*, 167:295–326, 2007.

[94] D. V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[95] D. V. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

[96] H. Petyt and D. Spriano. Unbounded domains in hierarchically hyperbolic groups. *arXiv:2007.12535*, 2020.

[97] K. Rafi. A combinatorial model for the Teichmüller metric. *Geom. Funct. Anal.*, 17(3):936–959, 2007.

[98] J. Russell. Extensions of multicurve stabilizers are hierarchically hyperbolic. *arXiv:2107.14116*, 2021.

[99] J. Russell. From hierarchical to relative hyperbolicity. *Int. Math. Res. Not.*, 2022(1):575–624, 2022.

[100] L. Sabalka. Embedding right-angled Artin groups into graph braid groups. *Geom. Dedicata*, 124:191–198, 2007.

[101] Z. Sela. Acylindrical accessibility for groups. *Invent. Math.*, 129(3):527–565, 1997.

[102] Z. Sela. Diophantine geometry over groups VI: the elementary theory of a free group. *Geom. Funct. Anal.*, 16(3):703–730, 2006.

[103] A. Sisto. 3–manifolds have unique asymptotic cones. *arXiv:1109.4674*, 2011.

[104] A. Sisto. Projections and relative hyperbolicity. *Enseign. Math.*, 59(1/2):165–181, 2013.

[105] A. Sisto. Contracting elements and random walks. *J. Reine Angew. Math.*, 2018(742):79–114, 2018.

[106] J. Świątkowski. Estimates for homological dimension of configuration spaces of graphs. *Colloq. Math.*, 89(1):69–79, 2001.

[107] W. Thurston. The geometry and topology of three-manifolds. *Princeton Univ. Math. Dept. Notes*, 1979.

[108] W. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[109] D. Wise. *The Structure of Groups with a Quasiconvex Hierarchy*, volume 366 of *Annals of Mathematics Studies*. Princeton University Press, 2021.